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Eulerian direct interaction applied to the statistical motion of particles in a turbulent fluid

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Eulerian direct interaction is used to close Liouville's equation central to the transport of particles in a turbulent fluid where the dominant drag force is derived from the particle and local fluid velocities. The reliability of the equation is then tested by comparison of solutions with those of a computer simulation of particle motion with Stokes drag in a random velocity field. Using an empirical drag law accurate for particle Reynolds numbers up to 500, formulae are derived for the statistical moments central to particle dispersion for a weak drag force operating in a Gaussian isotropic stationary velocity field. These show for instance that the long time particle diffusion coefficient is in general greater than the equivalent value based on Stokes drag.

1. Introduction

In a previous paper (Reeks 1977) we considered the statistical motion of particles immersed in a turbulent fluid, for a drag force assumed linear in the relative velocity between particle and local fluid velocity. In this situation all statistical moments associated with the particle motion are linear functionals of the equivalent fluid point average evaluated along a particle trajectory. It was the evaluation of this latter quantity that formed the basis of the analysis using a method previously employed by Phythian (1975) in studying fluid point motion, based on a second approximation to an iterative solution to the equation of motion. The technique is however only tractable analytically for linear drag which has a somewhat limited range of applicability, being restricted to particles with Reynolds numbers very much less than unity. The formulation here was Lagrangian and the closure problem implicit in an Eulerian framework manifested itself in the problem of finding a relationship between Lagrangian variables and Eulerian variables which formed the natural description of the turbulent velocity field.

We would like in this paper to reconsider an Eulerian formulation in an attempt to derive a transport equation for an ensemble of particles initially prepared in a known statistical manner. We adopt an Eulerian formulation firstly because it represents a framework in which we can more readily handle more general drag forces and secondly the closure problem we shall encounter is similar to that encountered in hydrodynamic turbulence but with none of the very formidable difficulties associated with an explicitly dynamically nonlinear equation of motion. Indeed the quest for a reliable closure approximation in turbulence has afforded us with closure schemes that are applicable to any general stochastic system.

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We shall restrict ourselves to drag forces that are functionally dependent upon the particle and fluid velocities but as such the precise dependence remains unspecified. We have done so as a mere matter of expediency and because such drag forces form a natural extension of the linear drag case we referred to earlier. Such forces are akin to the drag force generated in steady motion where the force aligns itself with the relative velocity between particle and fluid. In practical terms we are referring to the type of motion encountered in a turbulent gas where the high ratio of particle to fluid density precludes the influence of drag forces derivable from particle fluid accelerations. In addition we assume that we are dealing with low concentrations ensuring that particle–particle interactions together with any influence of the particles on the levels of turbulence can be neglected. In this sense a statistical average over an ensemble of realizable states of a single particle becomes an equivalent description.

Although from a practical aspect we are usually concerned with the dispersion of particle density in real space, the dynamical equations are such that real space alone is an incomplete framework in which to formulate the problem. For the particular drag force we are considering, the equations of motion are coupled equations in both particle velocity and position – what position a particle occupies is intimately bound up with its history in velocity space and vice versa. It is clear that a necessary starting point for an Eulerian formulation will be the Liouville equation for the particle density in the six-dimensional particle phase space for a single realization of the turbulence.

To be more explicit the stochastic process we are referring to is defined by a particle equation of motion of the form

$$\frac{d\mathbf{v}}{dt} = \beta\mathbf{\Psi}[\mathbf{v}, \mathbf{u}(\mathbf{x}, t)], \tag{1.1}$$

where $\mathbf{v}$ is the velocity of a particle at time t and position $\mathbf{x}$ in a turbulent velocity field for which $\mathbf{u}(\mathbf{x}, t)$ represents a single realization; $\mathbf{\Psi}$ is a prescribed function of $\mathbf{v}$ and $\mathbf{u}$ whose statistical behaviour is derived from the statistics of the turbulent velocity field. We define it such that when the particle Reynolds number is much less than 1 $\mathbf{\Psi} = \mathbf{u} - \mathbf{v}$. In this way β^{-1} is identical to the particle relaxation time for Stokes drag. For convenience we shall suppose that $\mathbf{v}$, $\mathbf{x}$, t are dimensionless variables by suitable scaling of the measured variables on the intensity v_0 and typical wavenumber k_0 of the turbulence. In this instance β becomes $(\tau_p k_0 v_0)^{-1}$, where τ_p is the particle relaxation time in normal units. By so doing, β is then a direct measure of the strength of the interaction and $\mathbf{\Psi}$ a measure of the departure from Stokes drag.

For a single realization of $\mathbf{u}(\mathbf{x}, t)$ Liouville's equation for the phase space density $\rho(\mathbf{v}, \mathbf{x}, t)$ is

$$\left(\frac{\partial}{\partial t} + L\right)\rho(\mathbf{v}, \mathbf{x}, t) = 0, \tag{1.2}$$

where L is the Liouville operator, explicitly

$$L = \mathbf{v}.\frac{\partial}{\partial \mathbf{x}} + \beta\frac{\partial}{\partial \mathbf{v}}.\mathbf{\Psi}. \tag{1.3}$$

An average of the solutions of this equation over all realizations of the turbulence represents a distribution $\bar{\rho}(\mathbf{v}, \mathbf{x}, t)$ which we may formally identify with the probability density in phase space for an ensemble of states generated in a prescribed statistical manner by the turbulent field. Our aim is to formulate an explicit equation for $\bar{\rho}(\mathbf{v}, \mathbf{x}, t)$ rather than solving equation (1.3) repeatedly many times and averaging.

Since equation (1.3) is explicitly linear in ρ,

$$\rho(\mathbf{v}, \mathbf{x}, t) = \int \hat{G}(\mathbf{v}, \mathbf{x}, t;\ \mathbf{v}', \mathbf{x}', t')\, \rho(\mathbf{v}', \mathbf{x}', t')\, d\mathbf{v}'\, d\mathbf{x}', \tag{1.4}$$

where $\hat{G}$ is the Green's function which satisfies

$$\left(\frac{\partial}{\partial t} + L\right) \hat{G}(\mathbf{v}, \mathbf{x}, t;\ \mathbf{v}', \mathbf{x}', t') = \delta(\mathbf{v}-\mathbf{v}')\, \delta(\mathbf{x}-\mathbf{x}')\, \delta(t-t'). \tag{1.5}$$

The fact that there is no reaction of ρ on $\mathbf{\Psi}$ means that $\hat{G}$ is a functional of $\mathbf{\Psi}$ and not of ρ, so that together with an assumed statistical independence of ρ and $\mathbf{\Psi}$ at some initial time t' it means

$$\bar{\rho}(\mathbf{v}, \mathbf{x}, r) = \langle \rho(\mathbf{v}, \mathbf{x}, t) \rangle = \int G(\mathbf{v}, \mathbf{x}, t;\ \mathbf{v}', \mathbf{x}', t')\, \langle \rho(\mathbf{v}', \mathbf{x}', t') \rangle\, d\mathbf{v}'\, d\mathbf{x}', \tag{1.6}$$

with

$$G(\mathbf{v}, \mathbf{x}, t;\ \mathbf{v}', \mathbf{X}', t') = \langle \hat{G}(\mathbf{v}, \mathbf{x}, t;\ \mathbf{v}', \mathbf{x}', t') \rangle.$$

So that in order to determine $\bar{\rho}$ it is sufficient to find an equation for G. If we decompose $\mathbf{\Psi}$ into an average part $\bar{\mathbf{\Psi}}$ and a fluctuating part $\tilde{\mathbf{\Psi}}$ we may write equation (1.5) more conveniently as

$$\left(\frac{\partial}{\partial t} + \bar{L}\right) \hat{G}(\mathbf{v}, \mathbf{x}, t;\ \mathbf{v}', \mathbf{x}', t') = -\beta \frac{\partial}{\partial \mathbf{v}} \cdot \tilde{\mathbf{\Psi}} \hat{G} + \delta(\mathbf{v}-\mathbf{v}')\, \delta(\mathbf{x}-\mathbf{x}')\, \delta(t-t'), \tag{1.7}$$

where the averaged Liouville operator $\bar{L}$ is given by

$$\bar{L} = \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{x}} + \beta \frac{\partial}{\partial \mathbf{v}} \cdot \bar{\mathbf{\Psi}},$$

so that the equation for G is naturally

$$\left(\frac{\partial}{\partial t} + \bar{L}\right) G(\mathbf{v}, \mathbf{x}, t;\ \mathbf{v}', \mathbf{x}', t') = -\beta \left(\frac{\partial}{\partial \mathbf{v}} \cdot \tilde{\mathbf{\Psi}} \hat{G}\right) + \delta(\mathbf{v}-\mathbf{v}')\, \delta(\mathbf{x}-\mathbf{x}')\, \delta(t-t'). \tag{1.8}$$

The closure problem implicit in (1.8) is almost self-evident. The term on the right-hand side is as yet undetermined. Using equation (1.7) to form an equation for $\langle \tilde{\mathbf{\Psi}} \hat{G} \rangle$ yields triple moments of the form $\langle \tilde{\mathbf{\Psi}} \tilde{\mathbf{\Psi}} \hat{G} \rangle$, and similarly the equation for third-order moments involves fourth-order moments and so on, so that a solution for $\bar{\rho}$ is bound up in an infinite hierarchy of moment equations. The closure approximation adopted in this paper is that provided by Kraichnan's Eulerian Direct Interaction approximation (EDI). To date EDI and the more sophisticated but less tractable Abridged Lagrangian-History Direct Interaction (ALHDI) have proved the most successful self-consistent approximations in closing the strongly interacting moment equations of hydrodynamic turbulence. This success seems quite remarkable when considering the formidable nature of the turbulence problem and the relative simplicity of the closed equations, which involve second-order moments only. But then their formulation does not rely in any way upon arbitrary truncation of the moment equations themselves. These facts alone make EDI and ALHDI very powerful and attractive techniques in approximating the statistical behaviour of dynamically simpler systems with an inherent closure problem. Indeed when applied to the problem of passive scalar diffusion in an isotropic random Gaussian and stationary velocity field, EDI gives results in remarkably good agreement with those of a computer simulation (Kraichnan 1970*a*). This is of course particularly encouraging since passive scalar diffusion represents a special case of the systems studied here, namely the case of $\beta \to \infty$, when fluid and particle motions are identical. Furthermore it probably represents the most severe test of the reliability of the approximation for this class of systems. However it must be admitted that the case of passive scalar diffusion in helical turbulence (Kraichnan 1977*a*) suggests that this good agreement may be limited to isotropic situations.

There are however more profound reasons which make EDI more acceptable and more appealing than any other closure approximation. In his remarkable paper on the dynamics of nonlinear stochastic systems Kraichnan (1961) showed that EDI is an exact closure to a particular model of the dynamic system which he has called the Random Coupling Model (RCM) and consequently has certain consistency properties lacking in other approximations. In other words we would expect a greater degree of physical realizability with EDI than with other approximations which do not have this model relationship with the real system. On may see EDI as the simplest truncation of a renormalized expansion of some averaged Eulerian property common to a class of systems containing the real system but it would seem that it is only this term in the infinity of terms that corresponds to a model dynamics. However this is not meant to imply that EDI is the only closure for which a model is available with its guarantee of realizability. In the case of turbulence for instance Kraichnan (1971) has constructed an 'almost Markovian Galilean invariant' model which has a greater degree of realizability than EDI. The unique quality of EDI in model dynamics of statistical systems is its implementability: one can always set up a renormalized expansion.

Renormalization represents a formally correct and elegantly simpler procedure in which we can generate EDI rather than considering the RCM equivalent, and we shall use this fact to set up a renormalized expansion for the term on the right-hand side of equation (1.8). For the RCM equivalent we refer the reader to Kraichnan's original paper and to the application of EDI to a Vlasov plasma (Kraichnan 1967), where the electric field is a direct counterpart of $\mathbf{\Psi}$, in the stochastic acceleration problem. In future for the sake of brevity and convenience we shall assume a degree of familiarity with both these papers.

2. EDI in relation to renormalized perturbation theory

It is more revealing to write equation (1.7) in matrix shorthand notation, namely

$$\mathbf{G}^{(0)-1}\hat{\mathbf{G}} = -l\hat{\mathbf{G}} + \mathbf{I}, \tag{2.1}$$

where l is equivalent to $(\beta\, \partial/\partial \mathbf{v}).\tilde{\mathbf{\Psi}}$, $\mathbf{I}$ the unit operator, and $\mathbf{G}^{(0)}$ the solution of the linear equation

$$\mathbf{G}^{(0)-1}\mathbf{G}^{(0)} = \mathbf{I},$$

$G^{(0)-1}(\mathbf{v}, \mathbf{x}, t;\, \mathbf{v}', \mathbf{x}', t')$ being explicitly $\bar{L}(\mathbf{v}, \mathbf{x}, t)\, \delta(\mathbf{v}-\mathbf{v}')\, \delta(\mathbf{x}-\mathbf{x}')\, \delta(t-t')$. Equation

(2.1) gives

$$\hat{\mathbf{G}} = \mathbf{G}^{(0)} - \mathbf{G}^{(0)} l \hat{\mathbf{G}}. \tag{2.2}$$

Equation (2.2) forms the basis of a formal primitive perturbation series for $\hat{G}$ in terms of functional powers of $G^{(0)}$ and l, namely

$$\hat{\mathbf{G}} = \mathbf{G}^{(0)} - \mathbf{G}^{(0)} l \mathbf{G}^{(0)} + \mathbf{G}^{(0)} l \mathbf{G}^{(0)} l \mathbf{G}^{(0)} - \mathbf{G}^{(0)} l \mathbf{G}^{(0)} l \mathbf{G}^{(0)} l \mathbf{G}^{(0)} + \ldots. \tag{2.3}$$

Referring to $\hat{G}(\mathbf{v}, \mathbf{x}, t;\, \mathbf{v}', \mathbf{x}', t')$ by $\hat{G}(1,2)$ we may write equation (2.3) in a more expanded form as

$$\begin{aligned}\hat{G}(1,2) = G^{(0)}(1,2) - G^{(0)}(1,3) * l(3)\, G^{(0)}(3,2) + G^{(0)}(1,3) * l(3)\, G^{(0)}(3,4) \\ * l(4)\, G^{(0)}(4,2) - G^{(0)}(1,3) * l(3)\, G^{(0)}(3,4) * l(4)\, G^{(0)}(4,5) \\ * l(5)\, G^{(0)}(5,2) + \ldots,\end{aligned}$$

where * means a convolution over repeated labels. Averaging this equation yields a primitive perturbation expansion for $G(1,2)$ in $G^{(0)}$, assuming for convenience that the odd moments of l vanish, namely

$$\begin{aligned}G(1,2) = G^{(0)}(1,2) + \langle l(3)\, l(4)\rangle\, G^{(0)}(1,3) * G^{(0)}(3,4) * G^{(0)}(4,2) \\ + \langle l(3)\, l(4)\, l(5)\, l(6)\rangle\, G^{(0)}(1,3) * G^{(0)}(3,4) * G^{(0)}(4,5) \\ * G^{(0)}(5,6) * G^{(0)}(6,2) + \ldots,\end{aligned} \tag{2.4}$$

where the averaged operator $\langle l(3)\, l(4) \ldots\rangle$ acts only on identical labels to the left in each $G^{(0)}$. The serious objections to using (2.4) directly as a basis for closure have been stressed elsewhere (Kraichnan 1966). The simplest way to obtain renormalization if only the first few terms are required is to revert the development of G in terms of $G^{(0)}$ by iteration to yield an expansion for $G^{(0)}$ in terms of functional powers of G and l (Kraichnan 1977*b*).

The first two terms are explicitly

$$G^{(0)}(1,2) = G(1,2) - \langle l(3)\, l(4)\rangle\, G(1,3) * G(3,4) * G(4,2). \tag{2.5}$$

Now from (2.3)

$$\begin{aligned}\langle l(1)\, \hat{G}(1,2)\rangle = -\langle l(1)\, l(3)\rangle\, G^{(0)}(1,3) * G^{(0)}(3,2) \\ - \langle l(1)\, l(3)\, l(4)\, l(5)\rangle\, G^{(0)}(1,3) * G^{(0)}(3,4) * G^{(0)}(4,5) * G^{(0)}(5,2) \\ + \ldots.\end{aligned} \tag{2.6}$$

Substituting for $G^{(0)}(1,2)$ we have

$$\begin{aligned}\langle l(1)\, \hat{G}(1,2)\rangle = -\langle l(1)\, l(3)\rangle\, G(1,3) * G(3,2) \\ - [\langle l(1)\, l(3)\, l(4)\, l(5)\rangle - \langle l(1)\, l(5)\rangle \langle l(3)\, l(4)\rangle \\ - \langle l(1)\, l(3)\rangle \langle (l(4)\, l(5)\rangle]\, G(1,3) * G(3,4) * G(4,5) * G(5,2) \\ + \ldots.\end{aligned} \tag{2.7}$$

The first term in this series corresponds to the EDI approximation. Presented in this very formal way there is no reason why this renormalized expansion should give any better results than the primitive expansion, except perhaps that each term does imply classes of terms from all orders of the primitive expansion. Indeed it is likely to suffer from the same problem of secularity. Its meaning can be demonstrated more transparently when we consider it in relation to a class of systems suggested by equation (2.2), of the form

$$\mathbf{G}^{-1}\hat{\mathbf{G}}_\lambda = \mathbf{I} + (\lambda^2\Sigma_2 + \lambda^4\Sigma_4 + \ldots)\,\hat{\mathbf{G}}_\lambda - \lambda l\hat{\mathbf{G}}_\lambda \tag{2.8}$$

(see, e.g., Phythian 1978).

For $\lambda = 0$ we have a non-random statistically sharp system, $\mathbf{G}^{-1}\hat{\mathbf{G}}_0 = \mathbf{I}$ and, for $\lambda = 1$, we deliberately generate the actual system we are interested in, so that

$$\Sigma_2 + \Sigma_4 + \ldots = \mathbf{G}^{(0)-1} - \mathbf{G}^{-1}, \tag{2.9}$$

where each Σ is a non-random kernel. In propagator renormalization we arrange for the averaged Green's function to be the same in every system, i.e. independent of λ. Each Σ can be written as functional powers of G so that equation (2.9) represents a closed equation for G. The difference from primitive perturbation theory is that we perturb about a general non-random system so that there is greater freedom to relate this system to the random system of interest. It is a way of linearizing the nonlinear random effects of the original system in a way that is reflected in the behaviour of the non-random system. It is reasonable to suppose that the more the averaged behaviour of the nonlinear system, defined in terms of the statistical moments, has in common with the non-random system at $\lambda = 0$, the better the latter will represent the total behaviour of the original system. In propagator renormalization the non-random system here has a common response function G. In the more general theory of vertex renormalization there is complete freedom to specify the degree of similarity but the number of closed equations naturally increases. Within this general scheme we might make $\langle\tilde{\rho}(1)\,\tilde{\rho}(2)\rangle$ identical in either system and so on but the evidence from passive scalar diffusion suggests that truncation after the first term on the right-hand side of equation (2.7) will be a sufficient representation.

In terms of renormalization EDI corresponds to the non-random equation

$$G^{(0)-1} - G^{-1} = \Sigma_2, \tag{2.10}$$

where

$$\Sigma_2 = \langle l(1)\,l(2)\rangle\, G(1,2),$$

There is a marked similarity between the non-random system implied by equation (2.10) and the RCM, which for the case of turbulence has been referred to elsewhere (Kraichnan 1970*b*). The Green's function in the non-random system is naturally statistically sharp but this is also true of the response function of the collective systems in the RCM when the number of the realizations of the original system which linearly compound a collective system tends to infinity. This is perhaps not too surprising since the generation of each collective system is similar to an averaging process on the original system. Additionally the response function of the RCM before randomizing the couplings is equal to the response function of the original system, analogous to the Green's function for the $\lambda = 0$ system. The randomization of the couplings between the equation of motion of the collective system is such that the magnitude of the coupling remains unchanged but the phases of these interactions are changed randomly in such a manner as to retain essential consistency properties of the original dynamics. In the equivalent primitive perturbation expansion for $\langle lG\rangle$ only certain classes of interactions survive and these turn out to be equivalent to the closed form given by (2.10). Remarkably, all dependence on third and higher moments of l are eradicated.

It has previously been remarked (Kraichnan 1966) that a closure scheme may be regarded as physically acceptable if it can preserve sufficiently the basic boundedness and invariance of the true dynamics. In the real system the simplest integral conservation laws associated with equation (1.2) are

$$\frac{d}{dt}\int \bar{\rho}(\mathbf{v},\mathbf{x},t)\,d\mathbf{v}\,d\mathbf{x} = 0, \tag{2.11}$$

$$\frac{d}{dt}\int [\bar{\rho}^2(\mathbf{v},\mathbf{x},t)+\langle\tilde{\rho}^2(\mathbf{v},\mathbf{x},t)\rangle]\,d\mathbf{v}\,d\mathbf{x} = 0, \tag{2.12}$$

together with the energy relationship between the inertial and particle–fluid drag forces,

$$\frac{d}{dt}\int \left[\bar{\rho}\frac{v^2}{2}-\beta\mathbf{v}\,.\,\bar{\mathbf{\Psi}}\bar{\rho}-\beta\mathbf{v}\,.\,\langle\tilde{\rho}\tilde{\mathbf{\Psi}}\rangle\right]d\mathbf{v}\,d\mathbf{x} = 0. \tag{2.13}$$

All of these relationships survive in the RCM because they reflect unaveraged conservation properties built into the equation for model Green's functions.

Equation (1.2) however implies an important inequality, namely that, if $\rho(\mathbf{v},\mathbf{x},t)$ is everywhere positive at some initial value of t,

$$\rho(\mathbf{v},\mathbf{x},t) \geqslant 0, \tag{2.14}$$

for all $\mathbf{v}$, $\mathbf{x}$ at all subsequent values of t. The simplest statistical inequality associated with (2.14) is

$$\bar{\rho}(\mathbf{v},\mathbf{x},t) \geqslant 0. \tag{2.15}$$

There is no guarantee in the RCM that this inequality will be satisfied for all possible forms of the drag moment – the concept of a particle trajectory is totally lost in the 'scrambling' of the original dynamics to form the collective system. This is the one disturbing feature of RCMs though it would seem that it is an inequality less likely to be violated than in other closure schemes.

Written out in full, the EDI equation for this system is

$$\left(\frac{\partial}{\partial t}+\mathbf{v}\,.\,\frac{\partial}{\partial \mathbf{x}}+\beta\frac{\partial}{\partial \mathbf{v}}\,.\,\bar{\mathbf{\Psi}}\right)G(\mathbf{v},\mathbf{x},t;\,\mathbf{v}',\mathbf{x}',t')$$

$$= \beta^2\frac{\partial}{\partial v_i}\int_{t'}^{t}d\xi\int d\boldsymbol{\omega}\,d\mathbf{y}\,G(\mathbf{v},\mathbf{x},t;\,\boldsymbol{\omega},\mathbf{y},\xi)\frac{\partial}{\partial \omega_j}\langle\tilde{\Psi}_i(\mathbf{v},\mathbf{x},t)\,\tilde{\Psi}_j(\boldsymbol{\omega},\mathbf{y},\xi)\rangle\,G(\boldsymbol{\omega},\mathbf{y},\xi;\,\mathbf{v}',\mathbf{x}',t'). \tag{2.16}$$

The final velocity distribution achieved by the ensemble is seen here on a macroscale as arising from a precise balance between the convection current $\bar{\mathbf{\Psi}}\rho$ and the diffusion current

$$-\int_{t'}^{\infty}d\xi\int d\boldsymbol{\omega}\,d\mathbf{y}\,d\mathbf{x}\,G(\mathbf{v},\mathbf{x},t;\,\boldsymbol{\omega},\mathbf{y},\xi)\frac{\partial}{\partial \omega_j}\langle\Psi_i(\mathbf{v},\mathbf{x},t)\,\Psi_j(\boldsymbol{\omega},\mathbf{y},\xi)\rangle\,\rho(\boldsymbol{\omega},\mathbf{y},\xi),$$

which is not locally dependent upon the particle-phase space concentration as it would be for a gradient transport model.

As a basic test of the reliability of the EDI equation we shall solve it in situations where deficiencies in the model are most likely to show up. In this latter sense the detailed form of $\mathbf{\Psi}$ is not particularly important, only the strength and time scales of the interaction. The simplest possible interaction to take is Stokes drag in fields which are homogeneous and stationary, i.e.

$$\boldsymbol{\Psi} = \mathbf{u} - \mathbf{v}, \tag{2.17}$$

forming an accurate description of motion for particle Reynolds numbers $\ll 1$.

3. The solution for linear drag

We wish to consider first motion with linear drag derivable from the simplest possible fluid velocity field for which an exact solution is known – namely a velocity field in which the velocity is constant in each realization but randomly distributed from one realization to the next. It is equivalent to turbulent motion in which the time scale of the fluid motion is infinite or at least very large compared with the relaxation time of the particle. Explicitly

$$\langle \tilde{\Psi}_i(\mathbf{v}, \mathbf{x}, t)\, \tilde{\Psi}(\boldsymbol{\omega}, \mathbf{y}, \xi) \rangle = \delta_{ij}, \tag{3.1}$$

and for convenience we shall set $\langle u_i \rangle = 0$.

It is clear that as far as the real system is concerned the particle will feel a constant drag during t' to t for each realization of $\mathbf{u}(\mathbf{x}, t)$. As far as the field is concerned we need only specify the probability density $P(u)$ of fluid velocity $\mathbf{u}$ at any time or place. Now the velocity $\mathbf{v}$ of a particle at time t, acted upon by a constant $\mathbf{u}$, starting with velocity $\boldsymbol{\omega}$ at time ξ is simply

$$\mathbf{v} = \boldsymbol{\omega}\, e^{-\beta(t-\xi)} + \mathbf{u}(1 - e^{-\beta(t-\xi)}), \tag{3.2}$$

for which the velocity distribution $\hat{G}(\mathbf{v}, t;\, \boldsymbol{\omega}, \xi)$ is

$$\hat{G}(\mathbf{v}, t;\, \boldsymbol{\omega}, \xi) = e^{\beta(t-\xi)}\, \delta(\mathbf{v}\, e^{\beta(t-\xi)} - \boldsymbol{\omega} - \mathbf{u}(e^{\beta(t-\xi)} - 1)). \tag{3.3}$$

$G(\mathbf{v}, t;\, \boldsymbol{\omega}, \xi)$ is thus precisely given by

$$G(\mathbf{v}, t;\, \boldsymbol{\omega}, \xi) = \int e^{\beta(t-\xi)}\, \delta(\mathbf{v}\, e^{\beta(t-\xi)} - \boldsymbol{\omega} - \mathbf{u}(e^{\beta(t-\xi)} - 1))\, P(u)\, d\mathbf{u}. \tag{3.4}$$

Let us now compare this exact solution with the solution given by the EDI equation under these conditions. Clearly we may write $G(\mathbf{v}, t;\, \boldsymbol{\omega}, \xi)$ in the form $G(\mathbf{s}, t-\xi)$ with $\mathbf{s} = \mathbf{v} - \boldsymbol{\omega}\, e^{-\beta(t-\xi)}$. It is simpler to consider now the equation for the characteristic function of $\mathbf{s}$, rather than deal with G direct. Explicitly

$$\breve{G}(\mathbf{k}, t-\xi) = \langle \exp i\mathbf{k}.\mathbf{s} \rangle = \breve{P}(k(1 - e^{-\beta(t-\xi)})), \tag{3.5}$$

where $\breve{P}(k)$ is the characteristic function for $\mathbf{u}$.

Integrating the EDI equation over all $\mathbf{x}$, and taking the Fourier transform with respect to $\mathbf{v}$ gives

$$\frac{\partial \breve{G}}{\partial t} + \beta \mathbf{k}.\frac{\partial \breve{G}}{\partial \mathbf{k}} = -k^2 \int_0^t e^{-\beta(t-\xi)}\, \breve{G}(k(1 - e^{-\beta(t-\xi)}))\, \breve{G}(k(e^{-\beta(t-\xi)} - e^{-\beta t}))\, d\xi \tag{3.6}$$

where for convenience we have set $t' = 0$.

Replacing $\beta(t-\xi)$ by τ, we have as $t \to \infty$

$$\mathbf{k}.\frac{\partial \breve{G}}{\partial k} = -k^2 \int_0^\infty e^{-\tau}\, \breve{G}(ke^{-\tau})\, \breve{G}(k(1 - e^{-\tau}))\, d\tau. \tag{3.7}$$

This may be written as

$$\frac{\partial \breve{G}(k)}{\partial k} = -\int_0^k \breve{G}(k-z)\,\breve{G}(z)\,dz. \tag{3.8}$$

For which the solution is (Roberts 1961),

$$\breve{G}(k) = \frac{J_1(2k)}{k}, \tag{3.9}$$

giving for the equilibrium velocity distribution

$$G(v) = \begin{cases} \dfrac{1}{(2\pi)^2}[4-v^2]^{-\frac{1}{2}} & \text{if} \quad v \leqslant 2, \\ 0 & \text{if} \quad v > 2. \end{cases} \tag{3.10}$$

This solution would be consistent with our original dynamics if $P(\mathbf{u})$ was identical in form to the above.

It is significant that in the RCM equivalent of this system the precise shape of $P(u)$ is irrelevant to the final outcome of $G(v,t)$; indeed, all fields so long as $\mathbf{u}$ is constant for each realization will give the same unique distribution. In the true dynamics, however, the final equilibrium state depends strongly on the total form of $P(u)$ in that $G(v)$ and $P(v)$ are identical. This precise correspondence is lost in the RCM because of the imposed randomization of the couplings. A reduction in the effective time and length scales associated with the true dynamics has a similar effect: the dependence of $G(v)$ on higher moments of the field is strongly reduced to zero and the final state is influenced only by the form of the second moment of the field. It is reasonable to suppose that in the situation we have analysed the influence of the higher moments on the form of G is likely to be greatest, though for a $P(u)$ given by (3.10) this amounts to precisely zero. It is more likely in a real situation for $P(u)$ to be close to Gaussian. The distribution in (3.10) is, however, physically acceptable – it is everywhere non-negative and contained within a non-zero volume of velocity space. These are important features of the RCM which do not survive in any finite-order perturbation theory or in quasi-linear theory. The sharp cusp and cut-off at $v = 2$ are however unattractive features though one might reasonably expect the cusp to relax as the correlation time associated with the second moment becomes comparable with the relaxation time of the particle (Kraichnan 1967). A comparison between (3.10) and a Gaussian velocity field (Roberts 1961) gives identical second moments, the fractional differences in the 4th, 6th and $2n$th moments being $\frac{1}{3}$, $\frac{2}{3}$ and $[1-2^n/(n+1)!]$ respectively.

The interesting feature of the system we have just analysed is that in terms of establishing an equilibrium velocity distribution the conditions are identical to particle motion coincident with that of the fluid, i.e. $\beta \rightarrow \infty$. It is evident that, in this instance, the velocity distribution will be established in an infinitely short time so that on the time scale of the particle motion the fluid will appear perfectly correlated. The distribution of velocities turns out to be identical in form to the solution of the EDI equation for passive scalar diffusion with short diffusion times (Roberts 1961). This is perhaps not surprising though it is significant that this result is unique to EDI with a linear drag law whereas in reality, so long as the drag is of the form $\Phi(|\mathbf{u}-\mathbf{v}|)(\mathbf{u}-\mathbf{v})$ with Φ always positive, it must always be true. For real times greater than the correlation time of the turbulence the form of the fluid velocity correlation is effective in controlling only the dispersion of particles in real space. It is instructive at this stage to formulate the diffusion equation in real space from the original EDI equation (2.16) for particles in stationary homogeneous turbulence for which $\beta \to \infty$.

For linear drag in a stationary fluid,

$$\langle \tilde{\Psi}_i(\mathbf{v},\mathbf{x},t)\,\tilde{\Psi}_j(\boldsymbol{\omega},\mathbf{y},\xi)\rangle = \beta^2\langle u_i(\mathbf{x},t)\,u_j(\mathbf{y},\xi)\rangle = \beta^2 R_{ij}(\mathbf{x}-\mathbf{y},t-\xi),$$

and the EDI equation (2.16) degenerates into

$$\left(\frac{\partial}{\partial t}+\mathbf{v}.\frac{\partial}{\partial x}-\beta\frac{\partial}{\partial \mathbf{v}}.\mathbf{v}\right)G(\mathbf{v},\mathbf{x},t;\,\mathbf{v}',\mathbf{x}',t')$$
$$=\beta^2\int_{t'}^{t}d\xi\int d\boldsymbol{\omega}\,d\mathbf{y}\,\frac{\partial}{\partial v_i}G(\mathbf{v},\mathbf{x},t;\,\boldsymbol{\omega},\mathbf{y},\xi)\,R_{ij}(\mathbf{x}-\mathbf{y},t-\xi)\,\frac{\partial}{\partial \omega_j}G(\boldsymbol{\omega},\mathbf{y},\xi;\,\mathbf{v}',\mathbf{x}',t').$$

Using integration by parts the right-hand side transforms to

$$-\beta^2\int_{t'}^{t}d\xi\int d\boldsymbol{\omega}\,d\mathbf{y}\,R_{ij}(\mathbf{x}-\mathbf{y},t-\xi)\,G(\boldsymbol{\omega},\mathbf{y},\xi;\,\mathbf{v}',\mathbf{x}',t')\frac{\partial^2}{\partial v_i\,\partial\omega_j}G(\mathbf{v},\mathbf{x},t;\,\mathbf{v}',\mathbf{x}',t').$$

We are interested in the case of $\beta\to\infty$, and values of $t-t' \gg 1/\beta$ (which is asymptotically satisfied by all real times as $\beta\to\infty$). Under these conditions

$$G(\mathbf{v},\mathbf{x},t;\,\mathbf{v}',\mathbf{x}',t') \to G(\mathbf{v};\,\mathbf{x}-\mathbf{x}';\,t-t').$$

If this however were identically true for all times $t' \leqslant \xi \leqslant t$, then the right-hand side would be identically zero. However with $\beta\to\infty$, for values of ξ such that $t-\xi \sim 1/\beta$ the fluid motion is perfectly correlated and we know from the previous example that in these circumstances $G(\mathbf{v},\mathbf{x},t;\,\boldsymbol{\omega},\mathbf{y},\xi)$ will behave as $G(\mathbf{s},\mathbf{r},t-\xi)$, where

$$\left.\begin{aligned}\mathbf{s} &= \mathbf{v}-\boldsymbol{\omega}\,e^{-\beta(t-\xi)},\\ \mathbf{r} &= \mathbf{x}-\mathbf{y}-\frac{1}{\beta}\,\boldsymbol{\omega}(1-e^{-\beta(t-\xi)}),\end{aligned}\right\} \tag{3.11}$$

$\mathbf{r}$ being obtained in a similar manner to $\mathbf{s}$ from the equation for particle displacement with a constant $\mathbf{u}$. Since the $\boldsymbol{\omega}$ integration is only significant for values $t-\xi \sim 1/\beta$, i.e. $\xi\to t$, we may legitimately write $G(\boldsymbol{\omega},\mathbf{y},\xi;\,\mathbf{v}',\mathbf{x}',t')$ as $G(\boldsymbol{\omega};\,\mathbf{y}-\mathbf{x}',\xi-t')$. Using these substitutions the equation for the characteristic function of $\mathbf{s}$ and $\mathbf{r}$, namely

$$\breve{G}(\mathbf{k},\mathbf{q},t-\xi) = \langle e^{i(\mathbf{k}.\mathbf{s}+\mathbf{q}.\mathbf{r})}\rangle,$$

as $\beta\to\infty$, can be shown to be

$$\frac{\partial}{\partial t}\breve{G}(\mathbf{k},\mathbf{q},t)+(\beta\mathbf{k}-\mathbf{q}).\frac{\partial}{\partial \mathbf{k}}\breve{G}(\mathbf{k},\mathbf{q},t)$$
$$=-\beta^2\int_0^t d\tau\int d\mathbf{p}\,k_i\Big(k_j e^{-\beta\tau}+\frac{q_j-p_j}{\beta}(1-e^{-\beta\tau})\Big)\breve{R}_{ij}(\mathbf{p},\tau)\,\breve{G}(\mathbf{k},\mathbf{q}-\mathbf{p},\tau)\,\breve{G}(0,\mathbf{q},t-\tau) \tag{3.12}$$

with $t'=0$ for convenience, and $\breve{R}_{ij}(\mathbf{p},t-\xi)$ the Fourier transform of R_{ij} with respect to $\mathbf{x}-\mathbf{y}$.

Now

$$\breve{G}(0,\mathbf{q},t) \equiv \langle \exp i\mathbf{q}.\mathbf{r}\rangle.$$

Putting $\mathbf{k} = 0$ in equation (3.12) we have

$$\frac{\partial}{\partial t}\breve{G}(0,\mathbf{q},t) = \mathbf{q}\cdot\frac{\partial}{\partial \mathbf{k}}\breve{G}(0,\mathbf{q},t). \tag{3.13}$$

Comparing the left-hand side of (3.13) with the right-hand side in equation (3.12) gives for $\beta \to \infty$

$$q_i\frac{\partial}{\partial k_i}\breve{G}(0,\mathbf{q},t) = -\int_0^t d\tau\int d\mathbf{p}\, q_i(p_j-q_j)\breve{R}_{ij}(\mathbf{p},\tau)\breve{G}(0,\mathbf{q},\mathbf{p},\tau)\breve{G}(0,\mathbf{q},t-\tau) \tag{3.14}$$

so that from (3.13) we have finally

$$\frac{\partial}{\partial t}\breve{G}(0,\mathbf{q},t) = -\int_0^t d\tau\int d\mathbf{p}\, q_i(p_j-q_j)\breve{R}_{ij}(\mathbf{p},\tau)\breve{G}(0,\mathbf{q}-\mathbf{p},\tau)\breve{G}(0,\mathbf{q},t-\tau), \tag{3.15}$$

which is the EDI equation for passive scalar diffusion in a stationary homogeneous field (Roberts 1961; Kraichnan 1970).

We now consider the general case of a finite β where we would expect a lack of coincidence between particle and fluid motion. This has been recently studied by Reeks (1977) and by Pismen & Nir (1978) and we shall use their results as a basis for comparison. Unfortunately, we have found this general case unamenable to exact solution. As an alternative we have used crude approximations for G that are equivalent to Reeks' first approximations to particle motion; these are then used in the diffusion term of the EDI equation to yield a soluble equation that still retains the essential character of EDI. We begin by replacing $G(\mathbf{v},\mathbf{x},t;\,\boldsymbol{\omega},\mathbf{y},\xi)$ by the zeroth-order approximation

$$G^{(0)}(\mathbf{v},\mathbf{x},t;\,\boldsymbol{\omega},\mathbf{y},\xi) = e^{3\beta(t-\xi)}\delta(\boldsymbol{\Omega}-\boldsymbol{\omega})\,\delta(\mathbf{Y}-\mathbf{y}), \tag{3.16}$$

where

$$\boldsymbol{\Omega} = \mathbf{v}\,e^{\beta(t-\xi)},$$

and

$$\mathbf{Y} = \mathbf{x}-\frac{\mathbf{v}}{\beta}(e^{\beta(t-\xi)}-1), \tag{3.17}$$

to obtain

$$\left(\frac{\partial}{\partial t}-\beta\frac{\partial}{\partial \mathbf{v}}\cdot\mathbf{v}+\mathbf{v}\cdot\frac{\partial}{\partial \mathbf{x}}\right)G(\mathbf{v},\mathbf{x},t;\,\mathbf{v}',\mathbf{x}',t')$$
$$= \beta^2\frac{\partial}{\partial v_i}\int_{t'}^{t} d\xi\, e^{3\beta(t-\xi)}R_{ij}\left(\frac{\mathbf{v}}{\beta}(e^{\beta(t-\xi)}-1),t-\xi\right)\frac{\partial}{\partial \Omega_j}G(\boldsymbol{\Omega},\mathbf{Y},\xi;\,\mathbf{v}',\mathbf{x}',t'). \tag{3.18}$$

It is interesting to note that (3.18) is identical to the quasi-linear approximation. We now replace $R_{ij}[(\mathbf{v}/\beta)(e^{\beta(t-\xi)}-1),t-\xi]$ by its value for zero motion ($\beta = 0$), namely $R_{ij}(0,t-\xi)$.

Now equations (3.17) define a transformation

$$\left.\begin{aligned}\boldsymbol{\Omega} &= \Theta(\mathbf{v},t-\xi),\\ \mathbf{Y} &= \Phi(\mathbf{v},\mathbf{x},t-\xi).\end{aligned}\right\} \tag{3.19}$$

Let us thus define variables

$$\left.\begin{aligned}\mathbf{V} &= \Theta(\mathbf{v},t-t'),\\ \mathbf{X} &= \Phi(\mathbf{v},\mathbf{x},t-t'),\end{aligned}\right\} \tag{3.20}$$

which imply an inverse transformation

$$\mathbf{v} = \Theta^{-1}(\mathbf{V}, t-t'); \quad \mathbf{x} = \Phi^{-1}(\mathbf{V}, \mathbf{X}, t-t'), \tag{3.21}$$

so that

$$\Theta\Theta^{-1} = 1, \quad \Phi\Phi^{-1} = 1.$$

We now define a function $\chi(\mathbf{V}, \mathbf{X}, t-t')$, such that

$$\chi(\mathbf{V}, \mathbf{X}, t-t') = e^{3\beta(t-t')} G(\mathbf{v}, \mathbf{x}, t-t'), \tag{3.22}$$

with $\mathbf{v}'$ and $\mathbf{x}'$ understood. By so doing, the resultant equation for χ is entirely local in $\mathbf{V}$ and $\mathbf{X}$ and devoid of any convection terms. Explicitly

$$\frac{\partial}{\partial t}\chi(\mathbf{V}, \mathbf{X}, t-t') = \beta^2 \frac{\partial}{\partial v_i}\int_0^{t-t'} d\xi\, R_{ij}(0, t-\xi)\left(e^{-\beta(t-\xi)}\frac{\partial}{\partial v_i} + \frac{1}{\beta}(1-e^{-\beta(t-\xi)})\frac{\partial}{\partial x_i}\right)\chi(\mathbf{V}, \mathbf{X}, \xi). \tag{3.23}$$

We now replace $\chi(\mathbf{V}, \mathbf{X}, \xi)$ by $\chi(\mathbf{V}, \mathbf{X}, t)$. There is no physical justification for this replacement at this stage – only with regard to the final solution. We have broken out of the model constraints to give, as we shall see, a form for G which is devoid of the cusp-like quality of G that is the solution of EDI as $\beta \to \infty$,

$$\left(\frac{\partial}{\partial t} - \beta\,\mathrm{div}_{\mathbf{v}}\,\mathbf{v} + \mathbf{v}.\frac{\partial}{\partial \mathbf{x}}\right) G(\mathbf{v}, \mathbf{x}, t;\, \mathbf{v}, \mathbf{x}', t')$$

$$= \beta^2 \frac{\partial}{\partial v_i}\int_0^{t-t'} e^{\beta(\xi-t)} R_{ij}(t-\xi)\, d\xi \frac{\partial G}{\partial v_j} + \beta\frac{\partial}{\partial v_i}\int_0^{t-t'} R_{ij}(t-\xi)\,(1-e^{-\beta(t-\xi)})\, d\xi \frac{\partial G}{\partial x_j}. \tag{3.24}$$

If we again define variables

$$\left.\begin{aligned} \mathbf{s} &= \mathbf{v} - \mathbf{v}'\, e^{-\beta(t-t')}, \\ \mathbf{r} &= \mathbf{x} - \mathbf{x}' - (v'/\beta)(1-e^{-\beta(t-t')}), \end{aligned}\right\} \tag{3.25}$$

the above equation is translationally invariant in six-space when expressed in these new variables. Forming the characteristic function of $\mathbf{s}$ and $\mathbf{r}$, we have

$$\frac{\partial \breve{G}}{\partial t} + (\beta\mathbf{k} - \mathbf{q}).\frac{\partial \breve{G}}{\partial \mathbf{k}} = -\breve{G}(\mathbf{k}, \mathbf{q}, t)\int_0^t [k_i k_j \beta^2 e^{-\beta\tau} + \beta k_i q_j (1-e^{-\beta\tau})]\, R_{ij}(0, \tau)\, d\tau \tag{3.26}$$

with $t' = 0$ for convenience.

It is interesting to note that (3.26) can be obtained from (3.12) by putting $p = 0$,

$$G(0, q, t-\tau) = 1.$$

The arguments leading to (3.12) are however only strictly valid for $\beta \to \infty$. It is difficult to see to what extent these arguments as they stand are valid approximations for arbitrary β. If (3.26), based on the approximations preceding it, is an acceptable first approximation to G we must conclude with hindsight that this must also mean that (3.12) is approximately valid for all β.

For isotropic turbulence the solution is a Gaussian of the form

$$G(k, q, t) = \exp\left[-\tfrac{1}{2}(\mu_{11}k^2 + 2\mu_{12}kq + \mu_{22}q^2)\right], \tag{3.27}$$

where $\mu_{11}, \mu_{12}, \mu_{22}$ are the second moments associated with r and s and are the solutions of

$$\left.\begin{aligned}\frac{1}{2}\frac{d\mu_{11}}{dt}+\beta\mu_{11} &= \beta^2\int_0^t e^{-\beta\tau}\, R(0,\tau)\, d\tau,\\ \frac{d\mu_{12}}{dt}+\beta\mu_{12}-\mu_{11} &= \beta\int_0^t (1-e^{-\beta\tau})\, R(\tau)\, d\tau,\\ \frac{1}{2}\frac{d\mu_{22}}{d\tau} &= \mu_{12},\end{aligned}\right\} \tag{3.28}$$

with $\mu_{12}(0) = \mu_{22}(0) = \mu_{11}(0) = 0$. The solutions are

$$\left.\begin{aligned}\mu_{11}(t) &= \beta\int_0^t R(0,\tau)\, e^{-\beta\tau}\, d\tau - \beta\, e^{-2\beta t}\int_0^t e^{\beta\tau} R(0,\tau)\, d\tau,\\ \mu_{12}(t) &= (1-e^{-\beta t})\int_0^t (1-e^{-\beta(t-\tau)})\, R(0,\tau)\, d\tau,\\ \mu_{22}(t) &= 2\int_0^t d\tau\, R(0,\tau)\left(t-\tau-\frac{1}{\beta}-\frac{e^{-\beta t}}{\beta}\cosh(\tau-t)+\frac{e^{-\beta t}}{\beta}\,(e^{\beta\tau}+1)\right).\end{aligned}\right\} \tag{3.29}$$

We note that, in the limit $t \to \infty$,

$$\left.\begin{aligned}\mu_{11} = \langle v_i^2\rangle &= \beta\int_0^\infty R(0,\tau)\, e^{-\beta\tau}\, d\tau,\\ \mu_{12} = \langle v_i x_i\rangle &= \int_0^\infty R(0,\tau)\, d\tau.\end{aligned}\right\} \tag{3.30}$$

These expressions are equivalent in form to the correct μ, where $R(0,\tau)$ is identified with the fluid Lagrangian velocity correlation $U(\tau)$ along a particle trajectory. For zero motion $R(0,\tau)$ and $U(\tau)$ are identical, but in general $R(0,\tau) \geqslant U(\tau)$, the difference only becoming significant at $\beta = 0$ (Reeks 1977). It would seem appropriate here to briefly mention the ALHDI approximation for this system since it bears some resemblance to equation (3.24). $R_{ij}(0, t-\xi)$ is replaced by the correlation coefficient $\langle u_i(\mathbf{x}, t)\, u_j(\mathbf{v}, \mathbf{x}, t|\xi)\rangle$, where the Lagrangian velocity field $\mathbf{u}(\mathbf{v}, \mathbf{x}, t|\xi)$ is the fluid velocity measured at time ξ along a particle trajectory which passed through $(\mathbf{v}, \mathbf{x})$ at time t. To close the equation we of course require an equation for this correlation coefficient which is obtained in a similar manner to the equation for G (see, e.g., the moment equation for the generalized electric field co-ordinate in the Vlasov plasma, Orzag 1968).

Returning to equation (2.16) we now replace G in the diffusion term for linear drag by the approximate form in (3.27), to give a second approximation for G, namely

$$\begin{aligned}&\left(\frac{\partial}{\partial t}+\mathbf{v}.\frac{\partial}{\partial \mathbf{x}}-\beta\,\mathrm{div}_{\mathbf{v}}\,\mathbf{v}\right)G(\mathbf{v},\mathbf{x},t;\,\mathbf{v}',\mathbf{x}',t')\\ &\quad= \beta^2\frac{\partial}{\partial v_i}\int_{t'}^t G_0\left(\mathbf{v}-\boldsymbol{\omega}\, e^{-\beta(t-\xi)};\ \mathbf{x}-\mathbf{y}-\frac{\boldsymbol{\omega}}{\beta}\,(1-e^{-\beta(t-\xi)})\right) R_{ij}(\mathbf{x}-\mathbf{y}, t-\xi)\\ &\qquad\times\frac{\partial}{\partial \omega_j}\, G_0\left(\boldsymbol{\omega}-\mathbf{v}'\, e^{\beta(\xi-t')};\ \mathbf{y}-\mathbf{x}'-\frac{\mathbf{v}}{\beta}\,(1-e^{-\beta(\xi-t')}\right),\end{aligned} \tag{3.31}$$

where G_0 refers to the approximate Green's function defined by (3.27).

Whereas in the first approximation the equation for G was translationally invariant in both $\mathbf{s}$ and $\mathbf{r}$, this is no longer true of this equation. In fact the form of $G(\mathbf{s}, \mathbf{r}, t-t')$ will depend on the initial $\mathbf{v}'$. By using the primitive perturbation expansion (2.3), this can be shown to be also true of the real $G(\mathbf{s}, \mathbf{r})$. Forming the characteristic function of $\mathbf{r}$ and $\mathbf{s}$ and setting $t' = 0$ for convenience

$$\left(\frac{\partial}{\partial t} + (\beta\mathbf{k} - \mathbf{q}) . \frac{\partial}{\partial \mathbf{k}}\right) \breve{G}(\mathbf{k}, \mathbf{q}, t) = -\beta^2 \int_0^t d\tau \int d\mathbf{p}\, k_i \left(k_j e^{-\beta\tau} + \frac{q_j - p_j}{\beta}(1 - e^{-\beta\tau})\right) \breve{R}_{ij}(\mathbf{p}, \tau)$$
$$\times \breve{G}_0(k, |\mathbf{q} - \mathbf{p}|, \tau)\, \breve{G}_0\left(\left|\mathbf{k}\, e^{-\beta\tau} + \frac{\mathbf{q} - \mathbf{p}}{\beta}(1 - e^{-\beta\tau})\right|, q, t - \tau\right)$$
$$\times \exp[-i\mathbf{p} . \mathbf{v}' e^{-\beta t}(1 - e^{\beta\tau})] \quad (3.32)$$

so that, unless $t \to \infty$, the process is no longer isotropic in $\mathbf{r}$ and $\mathbf{s}$ except for the trivial case of $\mathbf{v}' = 0$.

We now use this equation to generate the second moments of $\mathbf{s}$ and $\mathbf{r}$. For divergence-free flows we have

$$\left.\begin{aligned} &\left(\frac{d}{dt} + 2\beta\right) \langle s_i s_j \rangle = (1 + \delta_{ij}) \beta^2 \int_0^t d\tau e^{-\beta\tau} U_{ij}(\tau, t, \mathbf{v}'), \\ &\left(\frac{d}{dt} + \beta\right) \langle s_i r_j \rangle = \langle s_i s_j \rangle + \beta \int_0^t d\tau (1 - e^{-\beta\tau}) U_{ij}(\tau, t, \mathbf{v}'), \\ &\frac{d}{dt} \langle r_i r_j \rangle = 2 \langle s_i s_j \rangle, \end{aligned}\right\} \quad (3.33)$$

where

$$U_{ij}(t, \tau, \mathbf{v}') = \int d\mathbf{p}\, \breve{R}_{ij}(\mathbf{p}, \tau)\, G_0\left(-\frac{p}{\beta}(1 - e^{-\beta\tau}), 0, t - \tau\right)$$
$$\times G_0(0, -p, \tau) \exp\left[-i\mathbf{p} . \frac{\mathbf{v}'}{\beta} e^{-\beta t}(1 - e^{\beta\tau})\right]. \quad (3.34)$$

The explicit form for $U_{ij}(t, \tau, \mathbf{v}')$ given here is

$$U_{ij}(t, \tau, \mathbf{v}') = \int d\mathbf{p}\, \breve{R}_{ij}(\mathbf{p}, \tau) \exp -\left[i\mathbf{p} . \mathbf{v}' e^{-\beta t}(1 - e^{\beta\tau})\right.$$
$$\left. + \tfrac{1}{2} p^2 \left(\frac{(1 - e^{-\beta\tau})^2}{\beta^2} \mu_{11}(t - \tau) + \mu_{22}(\tau)\right)\right]. \quad (3.35)$$

It is interesting to compare this expression with the equivalent form used by Pismen & Nir based based on Corrsin's hypothesis, namely

$$U_{ij}(\tau) = \int d\mathbf{p}\, \breve{R}_{ij}(\mathbf{p}, \tau) \exp\{-[i\mathbf{p} . \mathbf{v}_0 \tau + \tfrac{1}{2} p^2 \langle r_i^2(\tau) \rangle]\}, \quad (3.36)$$

where $\mathbf{v}_0$ refers to some steady drift velocity imposed on the motion by some constant external force, e.g. gravity. In this respect the qualitative behaviour of a constant drift velocity is equivalent to the effect of an initial velocity $\mathbf{v}'$, except that the effect of the latter dies away to zero as the motion progresses. The presence of the term $\exp[-\tfrac{1}{2}p^2(1/\beta^2)(1 - e^{\beta\tau})^2 \mu_{11}(t - \tau)]$ in (3.35) is unique to EDI, and reduces the effective dispersion in relation to that implied by (3.36). Pismen & Nir use the actual $\mathrm{r}_i^2(\tau)$ in (3.36) to obtain a closed differential equation for $r_i^2(\tau)$ from equations (3.33). We could adopt the same procedure here by replacing $\mu_{11}(\tau)$ and $\mu_{22}(\tau)$ by $\langle r_i^2(\tau)\rangle$ and $\langle s_i^2(\tau)\rangle$ respectively. We shall not do so since the resulting equations are now integro-differential in form and less amenable to solution. The evidence suggests (Lundgren & Pointin 1976) that using zero motion approximations μ_{11}, μ_{22} will give values that are consistently close to the values obtained by complete solution of the closed equations in r_i^2 and s_i^2. They are evidently iteratively rapidly convergent.

An important test of the reliability of these approximations is provided by a comparison with a computer simulation in which we solve the equations of motion for a particle in a divergenceless, stationary, homogeneous, isotropic, Gaussian, velocity field. The procedure for generating such a field from an initial array of Gaussian random numbers is identical to that used by Kraichnan (1970*a*) in his simulation of fluid point motion. We shall adopt the procedure that produces a three-dimensional field with an $R_{ij}(\mathbf{x},\tau)$ of the form

$$R_{ij}(\mathbf{x},\tau) = \frac{1}{4\pi}\int d\mathbf{p}\left(\delta_{ij}-\frac{p_ip_j}{p^2}\right)\frac{E(p)}{p^2}\,e^{i\mathbf{p}\cdot\mathbf{x}}f(\tau) \tag{3.37}$$

explicitly, with a dimensionless energy spectrum of the form

$$E(p) = 16\left(\frac{2}{\pi}\right)^{\frac{1}{2}}p^4\exp-2p^2 \quad\text{and}\quad f(\tau) = e^{-\frac{1}{2}\tau^2}. \tag{3.38}$$

The form of $f(\tau)$ gives

$$\mu_{22}(\tau) = \left(\tau-\frac{1}{\beta}+\frac{e^{-\beta\tau}}{2}\right)\sqrt{(2\pi)}\,\mathrm{erf}\,\tau+2(e^{-\frac{1}{2}\tau^2}-1)$$
$$+\frac{1}{\beta}e^{-\beta\tau}\left(1-\frac{e^{-\beta\tau}}{2}\right)\sqrt{(2\pi)}\,e^{\frac{1}{2}\beta^2}\left(\mathrm{erf}\left(\frac{\tau-\beta}{\sqrt{2}}\right)+\mathrm{erf}\frac{\beta}{\sqrt{2}}\right)+\frac{1}{\beta}e^{\frac{1}{2}\beta^2}\sqrt{\frac{\pi}{(2)}}\left(\mathrm{erf}\frac{\beta}{\sqrt{2}}-\mathrm{erf}\left(\frac{\tau+\beta}{\sqrt{2}}\right)\right),$$
$$\mu_{11}(t-\tau) = \sqrt{\frac{\pi}{2}}\,e^{\frac{1}{2}\beta^2}\left(\mathrm{erf}\left(\frac{t-\tau+\beta}{\sqrt{2}}\right)-\mathrm{erf}\frac{\beta}{\sqrt{2}}-e^{2\beta(\tau-t)}\left(\mathrm{erf}\left(\frac{t-\tau+\beta}{\sqrt{2}}\right)+\mathrm{erf}\frac{\beta}{\sqrt{2}}\right)\right). \tag{3.39}$$

So that explicitly

$$U_{11}(\tau,t) = \frac{32}{\lambda^2\sqrt{\lambda}}\,e^{-\alpha^2/2\lambda},$$
$$U_{22}(\tau,t) = U_{33}(\tau,t) = \frac{16}{\lambda^3\sqrt{\lambda}}(2\lambda-\alpha^2)\,e^{-\alpha^2/2\lambda}, \tag{3.40}$$

with

$$\lambda = 4+\mu_{22}(\tau)+\frac{(1-e^{-\beta\tau})^2}{\beta^2}\mu_{11}(t-\tau) \tag{3.41}$$

and

$$\alpha = \frac{v'}{\beta}\,e^{-\beta t}(1-e^{\beta\tau}).$$

The axes are defined in terms of $\mathbf{v}'$, which is taken to be in the $i = 1$ positive direction. The particular particle studied in the simulation had a $\beta = 1$ and $\mathbf{v}' = 1$ and averages corresponding to those given in equations (3.33) were evaluated for 1000 realizations of the field. The effect of a $\mathbf{v}' = 1$ was found to be extremely weak both in the approximation and in the simulation so much so that the process could be regarded as isotropic in the space of $\mathbf{r}$ and $\mathbf{s}$ within the error of the simulation. The results are hence plotted as if the process were truly isotropic. The comparison between theory and

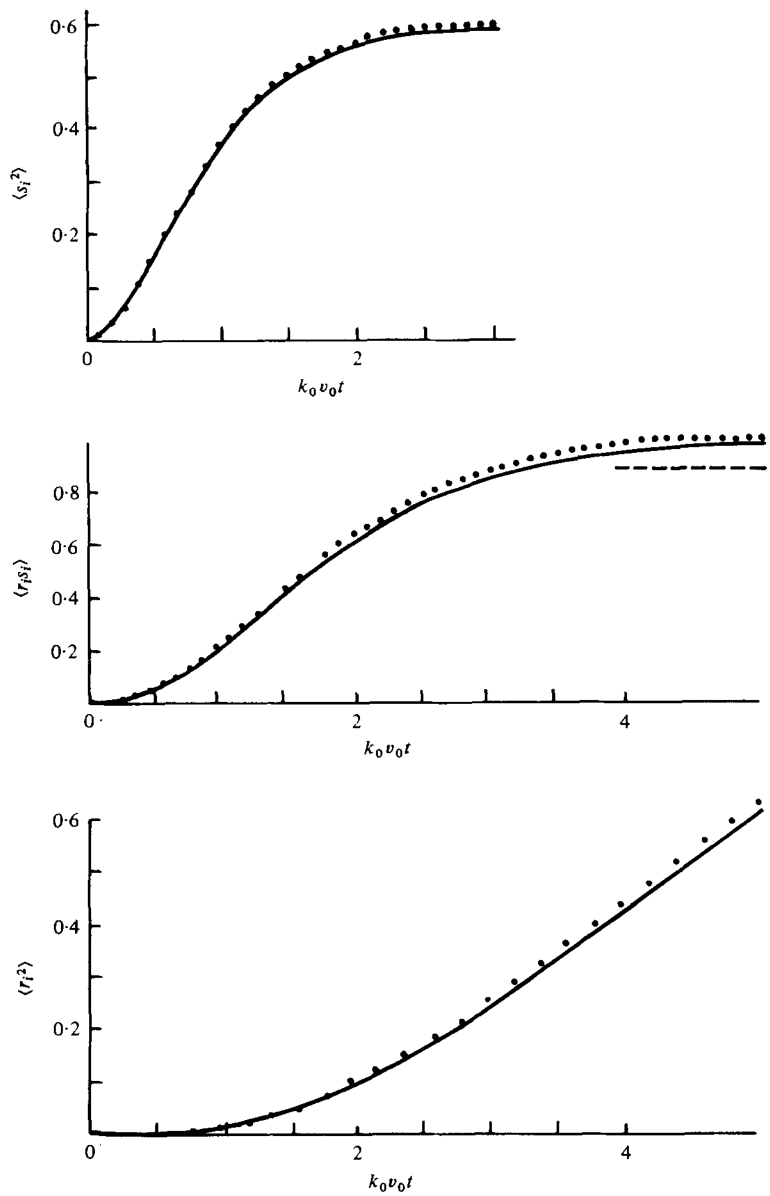

FIGURE 1. Comparison of theory with computer simulation. Curves obtained from equations (3.33) with $\beta = 1$, $\mathbf{v}' = 1$. Data points based on 1000 realizations of field. – – –, indicates the equivalent long time fluid point, diffusion coefficient. t is real time.

simulation, shown in figure 1, is remarkably good, well within the error of the simulation in some cases. It is interesting that the long time particle diffusion coefficient is greater than that for the fluid point (Kraichnan 1970*a, b*), a general feature of this system referred to previously (Reeks 1977). The comparison between this simulation and Pismen and Nir's approximation is also very good; the effect of the $\mu_{11}(t-\tau)$ term in U_{ij} appears to be very weak and likely to be even weaker as one moves towards higher or lower values of β. It does, however, improve the approximate behaviour of the

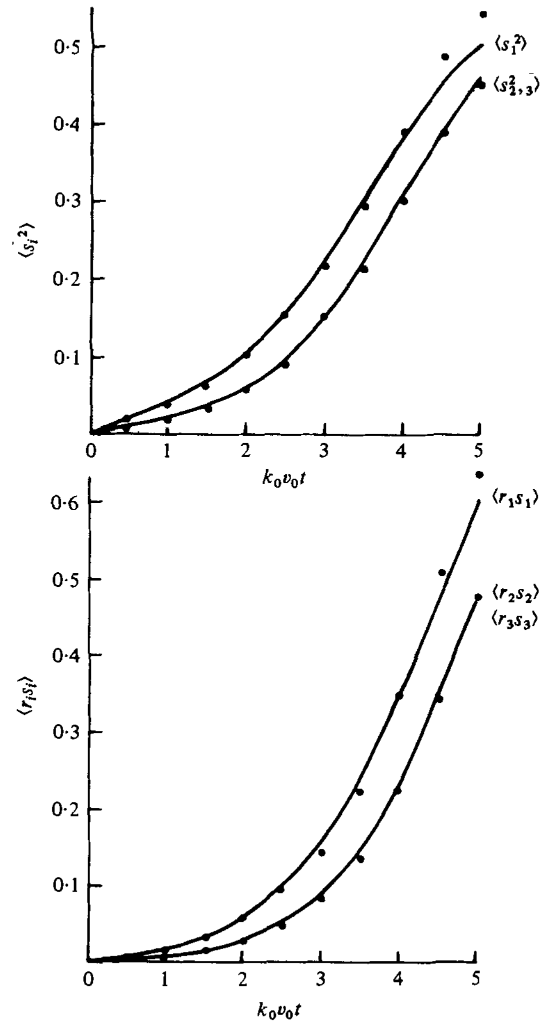

FIGURE 2. Comparison of theory with computer simulation. $\beta = 1$, $\mathbf{v}' = 100$.

system. The effect of an initial velocity upon the moments of $\mathbf{r}$ and $\mathbf{s}$ was found to be significant only at very large values of $\mathbf{v}'$, as shown in figure 2 where the difference between moments associated with $i = 1$ are markedly different from those for $i = 2, 3$ and significantly less than the equivalent values at $\mathbf{v}' \sim 0$, for $k_0 v_0 t \sim 1$. The most interesting result of this simulation is indicated in figure 3. In (*a*) and (*b*) the characteristic function of $\mathbf{s}$, $\check{G}(k,t)$ is evaluated at various values of time for values of k where G is significant. In (*c*), the characteristic function $\check{G}(k,q,t)$ is plotted as a function of k, for various values of q, as $t \rightarrow \infty$. In both figures the relevant data points are compared with a Gaussian whose second moments are those calculated by the approximation. In all cases the agreement is extremely good allowing one to conclude that, within the error of the simulation, the process is Gaussian, a result that we could not conclude from any formal theoretical procedure. Only in the case of $\beta = 0$, can we show this result to be exact.

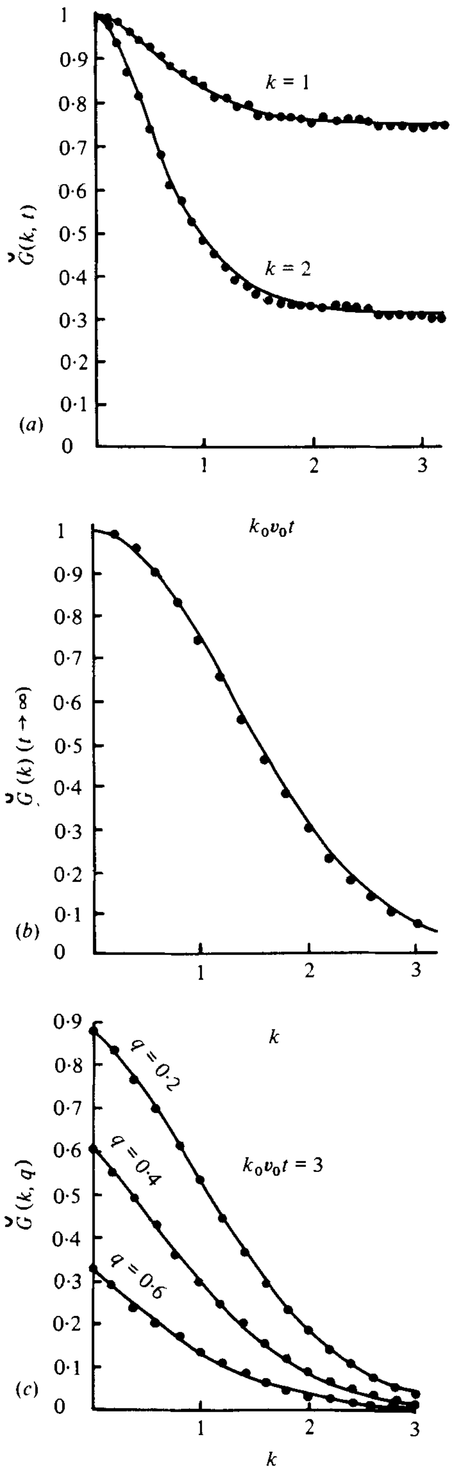

FIGURE 3. Comparison of characteristic functions with Gaussian based on equations (3.33), $\beta = 1$, $\mathbf{v}' = 1$.

4. The motion of very inert particles

The simplest and most instructive system in which we can consider the effects of the nonlinear terms in the drag force upon the statistical motion of particles is that corresponding to particle motion for which $\beta \ll 1$. This low value of β means that the particle is sufficiently inert to possess velocities small in relation to the turbulent intensity. The relevant fluid timescale is that associated with the eddy decay time implied by $f(\tau)$ in (3.37). We shall write τ as τ/T, where t is the real time, and define β^{-1} in units of T, the decay time associated with f. The assumption of $\mathbf{v} \ll 1$ means that we can write (2.16) in a quasi-linear form local in $\mathbf{v}$ and $\mathbf{x}$, similar to the equation for Stokes drag with $\beta \to 0$. Let us write $\mathbf{\Psi}$ more explicitly as

$$\mathbf{\Psi} = \Phi(Re\,|\mathbf{u}-\mathbf{v}|)\,(\mathbf{u}-\mathbf{v}). \tag{4.1}$$

Here Re is the 'turbulent' particle Reynolds number $v_0 d_p/\nu$, with v_0 the turbulent intensity, d_p the particle diameter and ν the kinematic viscosity. Φ is such that $\Phi(0) = 1$, and supposed to be in a form amenable to a power series expansion in $Re\,|\mathbf{u}-\mathbf{v}|$. Assuming an isotropic distribution for $\mathbf{u}$, symmetric about $\mathbf{u} = 0$, then, for small $\mathbf{v}$, we have

$$\begin{aligned}\beta\overline{\mathbf{\Psi}} &\simeq -\beta\langle \Phi(Re\,u) + \tfrac{1}{3}Re\,u\Phi'(Re\,u)\rangle\,\mathbf{v} \\ &= -\beta'\mathbf{v}\end{aligned} \tag{4.2}$$

and

$$\langle \tilde{\Psi}_i(\mathbf{v},\mathbf{x},0)\,\tilde{\Psi}_j(\boldsymbol{\omega},\mathbf{y},\xi)\rangle \simeq \langle \tilde{\Psi}_i(0,0,0)\,\tilde{\Psi}_j(0,0,\tau)\rangle, \tag{4.3}$$

which we shall abreviate to $\langle \tilde{\Psi}_i \tilde{\Psi}'_j\rangle$.

Replacing $G(\mathbf{v},\mathbf{x},t;\,\boldsymbol{\omega},\mathbf{y},\xi)$ in (2.16) by $\delta(\mathbf{v}-\boldsymbol{\omega})\,\delta(\mathbf{x}-\mathbf{y})$ consistent with $\beta \to 0$, we obtain for a stationary isotropic field

$$\left(\frac{\partial}{\partial t} - \beta'\,\mathrm{div}_{\mathbf{v}}\,\mathbf{v} + \mathbf{v}\,.\frac{\partial}{\partial x}\right) G(\mathbf{v},\mathbf{x},t;\,\mathbf{v}',\mathbf{x}',t') = \beta^2 \int_0^\infty \langle \tilde{\Psi}\tilde{\Psi}'\rangle\, d\tau\, \nabla_v^2 G(\mathbf{v},\mathbf{x},t;\,\mathbf{v}',\mathbf{x}',t') \tag{4.4}$$

for $t-t' \gg 1$. It is interesting to note that this equation can also be obtained by Fokker–Planck analysis (Sturrock 1966) and through the use of the Bourret smoothing approximation (Frisch 1968). Writing G as $G(\mathbf{s},\mathbf{r},t)$ we have $\check{G}(k,q,t)$ as Gaussian of the form

$$\check{G}(k,q,t) = \exp\left[-\tfrac{1}{2}(\mu_{11}k^2 + 2\mu_{12}kq + \mu_{22}q^2)\right]$$

with

$$\left.\begin{aligned}\mu_{11} &= (1-e^{-2\beta' t})\frac{\beta^2}{\beta'}\int_0^\infty \langle \tilde{\Psi}\tilde{\Psi}'\rangle\, d\tau, \\ \mu_{12} &= (1-e^{-\beta' t})^2\left(\frac{\beta}{\beta'}\right)^2\int_0^\infty \langle \tilde{\Psi}\tilde{\Psi}'\rangle\, d\tau, \\ \mu_{22} &= \left[t + \frac{2}{\beta'}e^{-\beta' t} - \frac{1}{2\beta'}e^{-2\beta' t} + \frac{3}{2\beta'}\right]\left(\frac{\beta}{\beta'}\right)^2\int \langle \tilde{\Psi}\tilde{\Psi}'\rangle\, d\tau.\end{aligned}\right\} \tag{4.5}$$

To evaluate $\langle \mathbf{\Psi}\rangle$ and $\langle \tilde{\Psi}\tilde{\Psi}'\rangle$ we shall adopt an empirical form for Φ reliable up to particle Reynolds numbers ~ 500, based on measurements of drag coefficients for spheres in steady flow (Serafini 1954), namely

$$\Phi(Re) = 1 + 0{\cdot}158 Re^{\frac{2}{3}} \tag{4.6}$$

and use a Gaussian for $\mathbf{u}$, necessarily of unit variance. In this instance

$$\beta' = \beta\left[1+0{\cdot}158\left(\frac{11}{9}.\frac{16^{\frac{1}{3}}\Gamma(11/6)}{\sqrt{\pi}}\right)Re^{\frac{2}{3}}\right] \tag{4.7}$$

$$\simeq \beta(1+0{\cdot}258248Re^{\frac{2}{3}}). \tag{4.8}$$

The evaluation of $\langle\tilde{\Psi}\tilde{\Psi}'\rangle$ is more complicated but straightforward. A two-point velocity correlation f for a time separation, τ, is consistent with a Gaussian bivariate distribution in $\mathbf{u}$ and $\boldsymbol{\omega}$ of the form

$$P(\mathbf{u},\boldsymbol{\omega}) = \frac{(1-f^2(\tau))^{-\frac{3}{2}}}{8\pi^3}\exp\left\{-\frac{1}{2(1-f^2(\tau))}[u^2+\omega^2-2\mathbf{u}.\boldsymbol{\omega}f(\tau)]\right\} \tag{4.9}$$

where $\langle u_i\omega_j\rangle = \delta_{ij}f(\tau), \langle u_i^2\rangle = \langle\omega_i^2\rangle = 1$.

It is readily seen that, using (4.6) for Φ,

$$\langle\tilde{\Psi}\tilde{\Psi}'\rangle = (\langle u_1\omega_1\rangle + 0{\cdot}316Re^{\frac{2}{3}}\langle u^{\frac{2}{3}}u_1\omega_1\rangle + (0{\cdot}158)^2\langle u^{\frac{2}{3}}\omega^{\frac{2}{3}}u_1\omega_1\rangle Re^{\frac{4}{3}}), \tag{4.10}$$

where, in general,

$$\langle u^p\omega^q u_1\omega_1\rangle = \int d\boldsymbol{\omega}\,d\mathbf{u}\,u^p\omega^q u_1\omega_1 P(\mathbf{u},\boldsymbol{\omega}). \tag{4.11}$$

Now by writing $\mathbf{u}$, $\boldsymbol{\omega}$ in terms of spherical co-ordinates (u,θ_1,ϕ_1) and (ω,θ_2,ϕ_2) respectively, and performing the θ_1, ϕ_1, θ_2, ϕ_2 integrations one can, after some labour, arrive at a power series in f given by

$$\langle u^p\omega^q u_1\omega_1\rangle = \frac{16}{3\pi}2^{\frac{1}{2}(p+q)}\sum_{m=0}^{\infty}A(m,p,q)f^{2m+1} \tag{4.12}$$

with

$$A(m,p,q) = \sum_{n=0}^{m}2^n\frac{\Gamma[n+\frac{1}{2}(p+5)]\,\Gamma[n+\frac{1}{2}(q+5)]}{(2n+3)!!\,n!}\binom{\frac{1}{2}(p+q+5)}{m-n}(-1)^{m-n}, \tag{4.13}$$

m and n both integer. Depending upon p and q, the terms in (4.12) are either zero after the first few terms or the series rapidly converges. The moments relevant to (4.10) are

$$\langle u^{\frac{2}{3}}u_1\omega_1\rangle = \frac{4}{3}\frac{2^{\frac{1}{3}}}{\sqrt{\pi}}\Gamma\left(\frac{17}{6}\right)f,$$

precisely, and

$$\langle u^{\frac{2}{3}}\omega^{\frac{2}{3}}u_1\omega_1\rangle = \frac{16}{3\pi}2^{\frac{2}{3}}\Gamma^2(\tfrac{5}{6})\,(0{\cdot}77803f+0{\cdot}024502f^3+0{\cdot}00045175f^5+0{\cdot}0001460f^7+\ldots). \tag{4.14}$$

Using the form of $f(\tau)$ given by (3.38) we have

$$\int_0^{\infty}\langle\tilde{\Psi}\tilde{\Psi}'\rangle d\tau \simeq (1{\cdot}2533+0{\cdot}64733Re^{\frac{2}{3}}+0{\cdot}08578Re^{\frac{4}{3}}). \tag{4.15}$$

It is interesting to quote the spatial diffusion coefficient, ϵ_{ii}, and particle mean square velocity compared to that of the fluid, $\langle v_i^2\rangle$, when the velocity distribution has reached equilibrium,

$$\epsilon_{ii}(\infty) = Tv_0^2\frac{1{\cdot}2533+0{\cdot}64733Re^{\frac{2}{3}}+0{\cdot}08578Re^{\frac{4}{3}}}{(1+0{\cdot}258248Re^{\frac{2}{3}})^2}, \tag{4.16}$$

$$\langle v_i^2(\infty)\rangle = \frac{T}{\tau_p}\left(\frac{1\cdot2533 + 0\cdot64733Re^{\frac{2}{3}} + 0\cdot08578Re^{\frac{4}{3}}}{1+0\cdot258248Re^{\frac{2}{3}}}\right). \tag{4.17}$$

We note again that the long time diffusion coefficient is independent of inertia and greater than the equivalent fluid point diffusion coefficient. Furthermore, it is, in general, greater than the equivalent value based on Stokes drag. The Reynolds number is seen here as controlling the effect of higher-order terms which are additive to the overall particle timescale, and hence to an increasing particle diffusion coefficient. As a practical illustration (4.16), with $Re = 200$, gives a value for ϵ_{ii} which $\sim 1\cdot5$ of that given by Stokes drag.

5. Concluding remarks

The basis of this paper has been to present a general transport equation to describe the behaviour of particles in a turbulent fluid, originating from a closure scheme based on EDI. It was limited to a drag law dependent upon the particle–fluid relative velocity, though this dependence was purely arbitrary and the character of the turbulence not necessarily stationary or homogeneous. For simplicity the specific examples we have used for solution have involved linear drag in some way or other and degenerate forms of turbulence. To handle more general nonlinear drag laws where the overall motion is not quasi-linear in particle velocity would require an explicit calculation of $\langle\tilde{\Psi}'\tilde{\Psi}''\rangle$ as a function of particle velocity. This can readily be achieved for motion restricted to one dimension but is considerably more difficult when the extension is made to three dimensions. What is clear is that the closure term would involve $f(v)\,\partial^2/\partial v^2$ which would enhance asymptotic diffusion based on purely Stokes' drag but restrict it to the quasi-linear form derived in (4.16) as $\beta \rightarrow 0$.

A noticeable deficiency in EDI is that in the limit of $\beta \rightarrow \infty$ the resulting distributions depend upon the form of drag law. It would be interesting to see to what extent this depends upon the fact that we have used an Eulerian rather than a Lagrangian closure. Although EDI yields a single closed equation, the fact that it involves terms which are functionals of the required phase space distribution has presented problems in obtaining analytic solutions and we have been forced to make initial approximations for the closure term except in trivial cases. We noted the resemblance between the approximate equation (3.24) on which the Stokes' law analysis was based and ALHDI. Indeed this equation is more logically a first approximation to ALHDI than EDI, and the basis of an iterative solution.

What has emerged as a general property of this type of motion as opposed to that of a passive scalar is that we cannot solve for the particle density in real space without knowing its total phase space density even if the turbulence itself is statistically stationary.

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REFERENCES

FRISCH, U. 1968 Wave propagation in random media. In *Probabilistic Methods in Applied Mathematics*, vol. 1 (ed. A. T. Bharucha-Reid). Academic.

KRAICHNAN, R. H. 1961 *J. Math. Phys.* **2**, 124.

KRAICHNAN, R. H. 1966 *Dynamics of Fluids and Plasmas* (ed. S. I. Pai.), p. 239. Academic.

KRAICHNAN, R. H. 1970*a* *Phys. Fluids* **13**, 22.

KRAICHNAN, R. H. 1970*b* *J. Fluid Mech.* **41**, 189.

KRAICHNAN, R. H. 1971 *J. Fluid Mech.* **47**, 513.

KRAICHNAN, R. H. 1977*a* *J. Fluid Mech.* **81**, 385.

KRAICHNAN, R. H. 1977*b* *J. Fluid Mech.* **83**, 349.

KRAICHNAN, R. H. & ORSZAG, S. A. 1967 *Phys. Fluids* **10**, 1720.

LUNDGREN, T. S. & POINTIN, Y. B. 1976 *Phys. Fluids* **19**, 355.

ORZAG, S. A. 1968 *Proc. of Symp. on Turbulence of Fluids and Plasmas, Microwave Res. Inst. Symposia Series*, vol. XVIII, pp. 17–28. Brooklyn, N.Y.: Polytechnic Press.

PHYTHIAN, R. 1975 *J. Fluid Mech.* **67**, 145.

PHYTHIAN, R. 1978 The application of renormalized perturbation theory in turbulence and related problems. In *Conf. Mathematical Modelling of Turbulent Diffusion in the Environment, University of Liverpool* (to be published).

PISMEN, L. M. & NIR, A. 1978 *J. Fluid Mech.* **84**, 193.

REEKS, M. W. 1977 *J. Fluid Mech.* **83**, 529.

ROBERTS, P. H. 1961 *J. Fluid Mech.* **11**, 247.

STURROCK, P. 1966 *Phys. Rev.* **141**, 186.

SERAFINI, J. S. 1954 *N.A.C.A. Tech. Rep.* 1159.